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The invariant manifolds of systems with first integrals☆

V.D. Irtegov, T.N. Titorenko

Irkutsk, Russia

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In fond memory of Pavel Alekseyevich Kuz'min†

ABSTRACT

Several additional possibilities of the Routh–Lyapunov method for isolating and analysing the stationarity sets of dynamical systems admitting of smooth first integrals are discussed. A procedure is proposed for isolating these sets together with the first integrals corresponding to the vector fields for these sets. This procedure is based on solving the stationarity equations of the family of first integrals of the problem in part of the variables and parameters occurring in this family. The effectiveness of this approach is demonstrated for two problems in the dynamics of a rigid body.

1. Introduction

In a lecture at the first Chetayev conference[1], Kuz'min drew attention to the great unrealized possibilities of the Routh–Lyapunov method[2] for analysing systems of differential equations admitting of smooth first integrals. The active use and extension of this approach for the above-mentioned class of equations still continues (particularly, in mechanics) up to the present time.[3–5] In essence, the method reduces to the isolation and qualitative investigation of the sets (manifolds) of stationarity of the elements of the algebra of the first integrals of the problem. Sometimes, this is equivalent to a complete analysis of a system.[6] In the general case, attempts to carry out an exhaustive analysis of even a fully integrable specific mechanical system using the approach discussed here leads to the practically unrealizable problem of checking all the elements of the algebra of the first integrals for an extremum. As a rule, they are confined to the analysis of a linear combination of the basic integrals.

An extension to the Routh–Lyapunov technique of isolating the invariant manifolds (IMs) of dynamical systems (and their submanifolds), which deliver a fixed value to the first integrals of the problem, is proposed in this paper. We shall call these manifolds the invariant manifolds of the steady motions (IMSMs).[7] A procedure is proposed for isolating the IMSM together with the set of first integrals corresponding to the vector field for it. The procedure reduces to solving the equations for the stationarity of the family of first integrals of the problem with respect to part of the variables and parameters occurring in this family.

Since, in the case of several algebraic integrals, the standard technique of isolating the IMSM of a mechanical system now leads to the analysis of a degenerate system of non-linear algebraic equations with parameters, assertions which enable one, on a semiheuristic level, to facilitate the search for the solutions of these equations are useful. The following[8] is one such assertion.

Assertion 1. Suppose the system of differential equations

$$\dot{x}_i = X_i(t, x_1, \ldots, x_n), \quad i = 1, \ldots, n \tag{1.1}$$

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E-mail address: irteg@icc.ru (V.D. Irtegov).

† A hundred years have passed since the birth of Pavel Alekseyevich Kuz'min (16.11.1908 - 31.07.92), Doctor of Science in Physics and Mathematics, Professor, and Honoured Scientist and Technologist of the TASSR.
Kuz'min was one of the first pupils of N. G. Chetayev, who held the Chair of Theoretical Mechanics at the Kazan Aviation Institute from 1938 to 1975, and was recognised as Head of the Kazan Chetayev School of Mechanicians after Chetayev left for Moscow in 1940. From 1964, he was a member of the scientific council on the problem of "General Mechanics" in the Department of Mechanics and Control Processes of the Academy of Sciences of the USSR and a member of the Presidium of the Scientific-Methological Council on Theoretical Mechanics at the Ministry of Higher and Secondary Special Education of the USSR. He was an initiator and, right up to the end of his life, the organizer of the widely known Chetayev Conferences on the Theory of Stability and Analytical Mechanics: the first of these conferences was held in Kazan in 1962. His research in the field of mechanics and the stability of motion made a considerable contribution to the development of the ideas of A. M. Lyapunov and N. G. Chetayev. His pupils include an Academician and four Doctors of Science.

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has a family of smooth first integrals $V(t, x, \lambda)$, the partial derivatives of this family with respect to the variables x_i ($i = 1,\ldots, n$) can be represented in the form

$$\frac{\partial V}{\partial x_i} = \sum_{l=1}^{k} a_{il}(t,x,\lambda)\varphi_l(t,x,\lambda) + \sum_{l=1}^{k}\sum_{p=1}^{k} a_{ilp}(t,x,\lambda)\varphi_l(t,x,\lambda)\varphi_p(t,x,\lambda) + \ldots,$$
$$i = 1,\ldots,n$$

and the rank of the matrix $||a_{il}(t, x, \lambda)||$ is equal to k in the family of manifolds $\varphi_l(t, x, \lambda) = 0$ ($l = 1,\ldots, k$). The equations

$$\varphi_l(t,x,\lambda) = 0, \quad l = 1,\ldots,k \tag{1.2}$$

then define the family of IM of system (1.1) and the element of this family deliver a fixed value to the elements of the first integrals $V(t, x, \lambda)$. Consequently, Eq. (1.2) determine the family of IMSMs of system (1.1). An analysis of cases of a reduction in the rank of the matrix $||a_{il}(t, x, \lambda)||$ in manifold (1.2) enables one to isolate the submanifolds lying in it.

Below, we demonstrate the proposed approach to isolating the IM using the example of an analysis of Euler's equations in Clebsch–Tisserant–Brun, Poincaré–Zhukovskii, and Kirchhoff problems.

2. The Clebsch–Tisserant–Brun problem[9]

Consider Euler's equations in the previously adopted notation[10]

$$A_1\dot{\omega}_1 + (A_3 - A_2)(\omega_2\omega_3 - \gamma_2\gamma_3) = 0, \quad \dot{\gamma}_1 + \omega_2\gamma_3 - \omega_3\gamma_2 = 0\,(123) \tag{2.1}$$

Equations (2.1) admit of the following first integrals

$$2H = \Sigma_{(123)}A_1(\omega_1^2 + \gamma_1^2) = 2h$$
$$V_1 = \Sigma_{(123)}A_1\omega_1\gamma_1 = m, \quad V_2 = \Sigma_{(123)}(A_1^2\omega_1^2 - A_2A_3\gamma_1^2) = n, \quad V_3 = \Sigma_{(123)}\gamma_1^2 = c$$

In order to isolate the IM of Eq. (2.1) we make use of the linear combination of basic first integrals

$$2K = V_2 - 2\lambda H - 2\mu V_1 + \nu V_3 \tag{2.2}$$

where λ, μ and ν are certain constants.

We write the stationarity conditions for the family of first integrals K with respect to the variables of the problem

$$\frac{\partial K}{\partial \omega_1} = A_1[(A_1 - \lambda)\omega_1 - \mu\gamma_1] = 0\,(123)$$
$$\frac{\partial K}{\partial \gamma_1} = (\nu - \lambda A_1 - A_2A_3)\gamma_1 - \mu A_1\omega_1 = 0\,(123) \tag{2.3}$$

In the case when the relations between the parameters λ, μ and ν

$$\lambda\nu = \lambda(A_2A_3 + A_3A_1 + A_1A_2) - A_1A_2A_3$$
$$\lambda\mu^2 = (\lambda - A_1)(\lambda - A_2)(\lambda - A_3)$$

are satisfied, the stationarity equations take the form

$$\frac{\partial K}{\partial \omega_1} = A_1[(A_1 - \lambda)\omega_1 - \mu\gamma_1] = 0\,(123)$$
$$\lambda\mu\frac{\partial K}{\partial \gamma_1} = A_1(\lambda - A_2)(\lambda - A_3)[(A_1 - \lambda)\omega_1 - \mu\gamma_1] = 0\,(123)$$

and, consequently, by Assertion 1, they define the family of IM of the initial system of differential equations

$$(A_1 - \lambda)\omega_1 - \mu\gamma_1 = 0\,(123) \tag{2.4}$$

The elements of this family deliver a fixed value to the elements of the family of first integrals K (2.2). The family of IMSMs (2.4) has been determined earlier[7,10] and has been investigated for stability.

It should be noted that, if we put $\mu = 0$ in the family of integrals K and, instead of the integral K, take $\tilde{K} = -K$, then the six equations (2.3) when $\lambda = A_1$ reduce to three equations:

$$\frac{\partial \tilde{K}}{\partial \omega_2} = A_2(A_1 - A_2)\omega_2 = 0, \quad \frac{\partial \tilde{K}}{\partial \omega_3} = A_3(A_1 - A_3)\omega_3 = 0$$
$$\frac{\partial \tilde{K}}{\partial \gamma_1} = (A_1 - A_3)(A_1 - A_2)\gamma_1 = 0$$

and, by Assertion 1, they enable us to determine a further IM of the initial system of differential equations

$$\omega_2 = \omega_3 = \gamma_1 = 0 \tag{2.5}$$

The vector field in a given IMSM has the form

$$A_1\dot{\omega}_1 = (A_3 - A_2)\gamma_2\gamma_3, \quad \dot{\gamma}_2 = \omega_1\gamma_3, \quad \dot{\gamma}_3 = -\omega_1\gamma_2$$

If Eq. (2.1) are interpreted as the equations of motion of a rigid body with a fixed point (γ_1, γ_2, γ_3 are the direction cosines), then IM (2.5) will determine the pendulum oscillations of the body about the "horizontal" x axis. For such an interpretation, the constant of the integral V_3 (a Casimir function) must be assumed to be equal to unity and the configuration space of the problem can be assumed to be isomorphic with SO(3).

Using the integral $\tilde{K}$, it can be shown that IM (2.5) is stable subject to the condition that the x axis is the axis of the greatest moment of inertia of the body. Actually, in the case of the equations for the perturbed motion, the integral $\tilde{K}$ in the neighbourhood of IM (2.5) will appear as (deviations from ω_2, ω_3, γ_1 are denoted as ξ_2, ξ_3, η_1 below)

$$A_1\Delta\tilde{K} = (A_1 - A_2)(A_1 - A_3)\eta_1^2 + A_2(A_1 - A_2)\xi_2^2 + A_3(A_1 - A_3)\xi_3^2$$

Since this last quadratic form is of fixed sign with respect to the variables appearing in it when $A_1 - A_3 > 0$, $A_1 - A_2 > 0$, these conditions are sufficient for the stability of IM (2.5).

We will now attempt to separate out the IM for differential equations (2.1) corresponding to non-linear combinations of the basic integrals. For this purpose, we shall assume that, in the family of first integrals K (2.2), the parameters λ, μ and ν are constants of certain, as yet unknown, first integrals. We shall solve the system of stationarity equations (2.3) for the quantities ω_3, γ_1, λ, μ, ν noting that the system of equations is non-linear with respect to these unknowns.

As a result, one of the solutions of this problem will be

$$\begin{aligned}
&A_2\omega_2^2\omega_n\gamma_n + (A_n - A_{4-n})\omega_n\gamma_2^2\gamma_n + \omega_2\gamma_2((A_{4-n} - A_2)\gamma_n^2 - A_n\omega_n^2) = 0, \quad n = 1,3\\
&\lambda = \frac{1}{2(\gamma_2^2 - \omega_2^2)}[(A_1 + A_3)\gamma_2^2 - A_2\omega_2^2 - R(\omega_2^2)]\\
&\mu = -\frac{\omega_2}{2\gamma_2(\gamma_2^2 - \omega_2^2)}[(A_1 - 2A_2 + A_3)\gamma_2^2 + A_2\omega_2^2 - R(\omega_2^2)]\\
&\nu = \frac{1}{2\gamma_2^2}[(A_1A_2 + 2A_1A_3 + A_2A_3)\gamma_2^2 + A_2^2\omega_2^2 - A_2R(\omega_2^2)]
\end{aligned} \tag{2.6}$$

Here,

$$R(\omega_2^2) = \sqrt{4A_1(A_3 - A_2)\omega_2^2\gamma_2^2 + ((A_1 - A_3)\gamma_2^2 + A_2\omega_2^2)^2}$$

It can be verified that the first two equations in this solution determine the IM of the initial system of differential equations. The vector field in this IM appears in one of the maps as

$$\begin{aligned}
&\dot{\omega}_1 = \frac{(A_3 - A_2)\gamma_3}{2A_1A_3\gamma_2}[(A_1 + A_3)\gamma_2^2 - A_2\omega_2^2 + R(\omega_2^2)]\\
&\dot{\omega}_2 = \frac{(A_3 - A_1)\gamma_3\omega_1}{2A_3(A_3 - A_2)\gamma_2\omega_2}[(A_1 - A_3)\gamma_2^2 + (2A_3 - A_2)\omega_2^2 + R(\omega_2^2)]\\
&\dot{\gamma}_2 = -\frac{\gamma_2}{\omega_2}\dot{\omega}_2, \quad \dot{\gamma}_3 = \frac{\omega_1}{2(A_2 - A_3)\gamma_2}[(A_1 - 2A_2 + A_3)\gamma_2^2 + A_2\omega_2^2 + R(\omega_2^2)]
\end{aligned} \tag{2.7}$$

Equations (2.7) are obtained from the initial system (2.1) after eliminating the variables ω_3 and γ_1 using the equations determining the IM. The expressions for λ, μ and ν, contained in the solution which has been found, are the first integrals of differential equations (2.7).

It can be assumed that IM (2.6) delivers a fixed value to a certain "non-linear sheaf of the first integrals K (2.2)" where the expressions for the integrals corresponding to the constants λ, μ and ν are only known in the manifold (2.6). It can be checked that there is only one independent first integral among those which have been found above.

3. The Poincaré–Zhukovskii and Kirchhoff problems

We will now apply the approach demonstrated above to a dynamical system[11] and, introducing the notation

$$a = \alpha^2 + \beta^2, \ \zeta = \beta s_1 - \alpha s_2, \ \kappa = \alpha s_1 + \beta s_2, \ \rho_j = \alpha r_1 + \beta r_2 + j s_3, \ j = 0,1,2, \ \sigma = r_1 s_1 + r_2 s_2,$$

we will write the differential equations of this system in the form

$$\dot{s}_1 = \alpha r_1 s_2 - \alpha^2 r_2 r_3 - (\beta r_3 - s_2)(\beta r_2 + s_3)$$
$$\dot{s}_2 = a r_1 r_3 - \rho_0 s_1 + (\alpha r_3 - s_1)s_3, \quad \dot{s}_3 = (\beta r_1 - \alpha r_2)s_3$$
$$\dot{r}_1 = r_2\rho_2 - r_3 s_2 - x(a r_3 s_2 + \beta s_3^2)$$
$$\dot{r}_2 = r_3 s_1 - r_1\rho_2 + x(a r_3 s_1 + \alpha s_3^2), \quad \dot{r}_3 = r_1 s_2 - r_2 s_1 + x s_3\zeta \tag{3.1}$$

When $x > 0$ and $x < 0$, these equations can be considered as Euler equations in the Lie algebra so(4) and so(3,1). When $x = 1$, they can be interpreted as the Poincaré–Zhukovskii equations of a body with an ellipsoidal cavity filled with a fluid and, when $x = 0$, as the Kirchhoff equations of motion of a body in an ideal fluid.

Equations (3.1) have the following first integrals

$$2V_0 = (s_1^2 + s_2^2 + 2s_3^2) + 2\rho_0 s_3 - a r_3^2 = 2h$$
$$V_1 = \sigma + s_3 r_3 = c_1, \quad V_2 = x(s_1^2 + s_2^2 + s_3^2) + r_1^2 + r_2^2 + r_3^2 = c_2$$
$$2V_3 = \sigma(a\sigma + 2\kappa s_3) + s_3^2(s_1^2 + s_2^2 + \rho_1^2) + x s_3^2\zeta^2 = 2c_3$$

In order to find the IMSM of Eq. (3.1), we will form a whole linear sheaf of the integrals of this problem

$$K = \lambda_0 V_0 - \lambda_1 V_1 - \frac{\lambda_2}{2}V_2 - \lambda_3 V_3$$

and write the stationarity conditions for K with respect to the variables $s_1, s_2, s_3, r_1, r_2, r_3$

$$\frac{\partial K}{\partial s_1} = \lambda_0 s_1 - \lambda_1 r_1 - \lambda_3[a r_1\sigma + s_1 s_3(\alpha r_2 + \beta r_1) + s_2 s_3(2\alpha r_1 + s_3)] - x(\beta\lambda_3 s_3^2\zeta + \lambda_2 s_1) = 0$$
$$\frac{\partial K}{\partial s_2} = \lambda_0 s_2 - \lambda_1 r_2 - \lambda_3[a r_2\sigma + s_1 s_3(\alpha r_2 + \beta r_1) + s_2 s_3(2\beta r_2 + s_3)] - x(\alpha\lambda_3 s_3^2\zeta + \lambda_2 s_2) = 0$$
$$\frac{\partial K}{\partial s_3} = \lambda_0\rho_2 - \lambda_1 r_3 - \lambda_3[(r_1 s_1^2\kappa\sigma + s_3\rho_0^2 + s_3(s_1^2 + s_2^2 + 2s_3^2) + 3s_3^2\rho_0] - x s_3(\lambda_3\zeta^2 + \lambda_2) = 0$$
$$\frac{\partial K}{\partial r_1} = \alpha\lambda_0 s_3 - \lambda_1 s_1 - \lambda_2 r_1 - \lambda_3[a s_1\sigma + \alpha s_3^2\rho_0 + \alpha s_3(s_1^2 + s_2^2) + \beta\lambda_3 s_1 s_2 s_3] = 0$$
$$\frac{\partial K}{\partial r_2} = -\beta\lambda_0 s_3 + \lambda_1 s_2 + \lambda_2 r_2 + \lambda_3[a s_2\sigma + s_2 s_3\kappa + \beta s_3^2\rho_0 + \beta s_3^3] = 0$$
$$\frac{\partial K}{\partial r_3} = -(a\lambda_0 + \lambda_2)r_3 + \lambda_1 s_3 = 0 \tag{3.2}$$

We shall seek a solution of system (3.3), assuming that $\lambda_0, \lambda_1, \lambda_2, \lambda_3, s_1, r_3$ are unknown. Using the technique of Groebner bases[12] to solve systems of non-linear algebraic equations, it is fairly easy to obtain the following solution of system (3.2)

$$\beta r_3 - s_2 = 0, \quad \beta s_1 - \alpha s_2 = 0, \quad \lambda_0 = \lambda_3 s_3\rho_1,$$
$$\lambda_1 = -\frac{a\lambda_3 s_2\rho_1}{\beta}, \quad \lambda_2 = 0 \tag{3.3}$$

It is easily verified that the first two equations of (3.3) determine the IM of system of differential equations (3.1). The vector field in this invariant manifold is described by the equations

$$\dot{s}_2 = (\beta r_1 - \alpha r_2)s_2, \quad \dot{s}_3 = (\beta r_1 - \alpha r_2)s_3$$
$$\dot{r}_1 = r_2\rho_2 - \frac{1}{\beta}((ax+1)s_2^2 + \beta^2 s_3^2)), \quad \dot{r}_2 = \frac{\alpha s_2^2}{\beta^2} - r_1\rho_2 + \frac{\alpha x}{\beta^2}(a s_2^2 + \beta^2 s_3^2) \tag{3.4}$$

and the expressions for λ_0 and λ_1 are the first integrals of differential equations (3.4).

Choosing different combinations of the parameters λ_i and the variables r_j, s_j as the unknowns in the stationarity equations, it is possible to obtain the other solutions of these equations which will determine the IM of the initial differential equations. For example, assuming that $\lambda_1, \lambda_2, s_1, s_2, s_3, r_3$ are unknown, we obtain the following solution of Eq. (3.2)

$$r_3 = 0, \quad \lambda_3 s_1^2 r^2 - \lambda_0 r_1^2 = 0, \quad \lambda_3 s_2^2 r^2 - \lambda_0 r_2^2 = 0, \quad s_3 = 0$$
$$\lambda_1 = \frac{[a\lambda_3 r^2 - (ax-1)\lambda_0]\sqrt{\lambda_0\lambda_3}r}{\lambda_3 r^2 - \lambda_0 x}, \quad \lambda_2 = \frac{\lambda_0^2}{\lambda_0 x - \lambda_3 r^2}; \quad r^2 = r_1^2 + r_2^2$$

The first four equations determine the IM of system of differential equations (3.1). The vector field in this IM is described by the equations

$$\dot{r}_1 = r_2\rho_0, \quad \dot{r}_2 = -r_1\rho_0 \tag{3.5}$$

The expressions for λ_1 and λ_2 are the first integrals of differential equations (3.5).

In the case of differential equations (3.4) and (3.5), a problem can be set up for isolating IMSM, which we shall call second level IMSM.

4. Isolation of second-level invariant manifolds

We will now consider the problem of isolating the IM for differential equations (3.4).

Apart from the first integrals

$$\lambda_0 = \lambda_3 s_3 \rho_1, \quad \lambda_1 = -\frac{a\lambda_3 s_2 \rho_1}{\beta}$$

Eq. (3.4) possess the first integrals

$$\bar{V}_0 = s_3\rho_1 = \bar{h}, \quad \bar{V}_1 = s_2\rho_1 = \bar{c}_1$$
$$\bar{V}_2 = r_1^2 + r_2^2 + \frac{1}{\beta^2}((ax+1)s_2^2 + \beta^2 s_3^2) = \bar{c}_2$$
$$\bar{V}_3 = \rho_1^2(as_2^2 + \beta^2 s_3^2) = \bar{c}_3$$

which are obtained from the first integrals of the initial system of differential equations after eliminating the variables s_1 and r_3 from them using the first two equations of (3.3).

We select independent first integrals (such as, for example, $\bar{V}_2, \lambda_0, \bar{V}_3$) from the first integrals of system (3.4) and form the linear combination of them

$$\tilde{K} = \bar{V}_2 - \mu\lambda_0 - \nu\bar{V}_3 \tag{4.1}$$

Use of the technique described above for isolating IMs enabled us to obtain, for example, the following solution corresponding to integral (4.1)

$$\beta r_1 - \alpha r_2 = 0, \quad (ax+1)s_2^2 - f(s_3, r_2) = 0 \tag{4.2}$$

$$\mu = -\frac{2}{a\lambda_3}, \quad \nu = \frac{\beta^2(ax+1)}{a(ar_2^2 + s_3)^2} \tag{4.3}$$

Here,

$$f(s_3, r_2) = ar_2^2 + \beta(2r_2 - \beta x s_3)s_3$$

Equations (4.2) determine the IM of differential equations (3.4). The vector field in this IM is described by the equations

$$\dot{r}_2 = 0, \quad \dot{s}_3 = 0 \tag{4.4}$$

The expressions for ν and μ are first integrals of differential equations (4.4). It is obvious in this case that the integrals will be trivial by virtue of the fact that the equations themselves are trivial.

The IM of the initial differential equations, in which Eq. (4.2) and the equations

$$\beta^2(ax+1)s_1^2 - \alpha^2 f(s_3, r_2) = 0, \quad \beta^2(ax+1)r_3^2 - f(s_3, r_2) = 0$$

appear, will correspond to the second level IM found in the initial phase space. The vector field in this IMSM is described by Eq. (4.4).

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